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RESEARCH PAPER

HARDY-LITTLEWOOD, BESSEL-RIESZ, AND FRACTIONAL INTEGRAL OPERATORS IN ANISOTROPIC MORREY AND CAMPANATO SPACES

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Abstract

We analyze local (central) Morrey spaces, generalized local (central) Morrey spaces and Campanato spaces on homogeneous groups. The boundedness of the Hardy-Littlewood maximal operator, Bessel-Riesz operators, generalized Bessel-Riesz operators and generalized fractional integral operators in generalized local (central) Morrey spaces on homogeneous groups is shown. Moreover, we prove the boundedness of the modified version of the generalized fractional integral operator and Olsen type inequalities in Campanato spaces and generalized local (central) Morrey spaces on homogeneous groups, respectively. Our results extend results known in the isotropic Euclidean settings, however, some of them are new already in the standard Euclidean cases.

MSC 2010: Primary 22E30; Secondary 43A80

Key Words and Phrases: fractional integral operator, generalized local (central) Morrey space, Campanato space, Hardy-Littlewood maximal operator, Bessel-Riesz operator, Olsen type inequality, homogeneous Lie group

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1. **Introduction**

Consider the following Bessel-Riesz operators

$$I_{\alpha,\gamma}f(x)=\int_{\mathbb{R}^n}K_{\alpha,\gamma}(x-y)f(y)dy=\int_{\mathbb{R}^n}\frac{|x-y|^{\alpha-n}}{(1+|x-y|)^{\gamma}}f(y)dy, \qquad (1.1)$$

where $f\in L^p_{loc}(\mathbb{R}^n)$, $p\geq 1$, $\gamma\geq 0$ and $0<\alpha<n$. Here, $I_{\alpha,\gamma}$ and $K_{\alpha,\gamma}$ are called Bessel-Riesz operator and Bessel-Riesz kernel, respectively.

The boundedness of the fractional integral operators $I_{\alpha,0}$ on Lebesgue spaces was shown by Hardy and Littlewood in [24], [25] and Sobolev in [43]. In the case of $\mathbb{R}^n$, the Hardy-Littlewood maximal operator, the generalized fractional integral operators, which are a generalized form of the Riesz potential $I_{\alpha,0}=I_\alpha$, Bessel-Riesz operators and Olsen type inequalities are widely analysed on Lebesgue spaces, local (central) Morrey spaces and generalized local (central) Morrey spaces (see e.g. [1], [10], [29], [13], [11], [28], [30], [31], [23], [42], [26] and [27], as well as [6] for a recent survey). For some of their functional analytic properties, see also [7], [8] and references therein. We also refer [17], [18], [16], [5], [37] for analysis in local (central) Morrey spaces and generalized local (central) Morrey spaces, and [4], [3] in anisotropic local Morrey-type spaces.

In this paper we are interested in the boundedness of the Hardy-Littlewood maximal operator, Bessel-Riesz operators, generalized Bessel-Riesz operators, generalized fractional integral operators and Olsen type inequalities in generalized local (central) Morrey spaces on homogeneous Lie groups. The obtained results give new statements already in the Euclidean setting of $\mathbb{R}^n$ when we are working with anisotropic differential structure. Furthermore, even in the isotropic situation in $\mathbb{R}^n$, one novelty of all the obtained results is also in the arbitrariness of the choice of any homogeneous quasi-norm, and some estimates are also new in the usual isotropic structure of $\mathbb{R}^n$ with the Euclidean norm, which we will be indicating at relevant places.

Thus, we could have worked directly in $\mathbb{R}^n$ with anisotropic structure, but since the methods work equally well in the setting of Folland and Stein's homogeneous groups, we formulate all the results in such (greater) generality. In particular, it follows the general strategy initiated by their work, of distilling results of harmonic analysis depending only on the group and dilation structures: in this respect the present paper shows that the harmonic analysis on local (central) Morrey spaces largely falls into this category.

In turn, this continues the research direction initiated in [36] devoted to Hardy and other functional inequalities in the setting of Folland and Stein's [15] homogeneous groups. We also refer to recent papers [32], [33], [34], [35], [38], [39], [40] and [41] for discussions related to different functional inequalities with special as well as arbitrary homogeneous quasi-norms in different settings. Local (central) Morrey spaces for non-Euclidean distances find their applications in many problems, see e.g. [20, 21] and [22].

We also refer to the recent survey paper [2] for some inequalities in fractional calculus that are used in differential or integral equations, and refer to [9] for the boundedness of the Riesz fractional integration operator from a generalized Morrey space $L^{p,\phi}(\mathbb{R}^n)$ to a certain Orlicz-Morrey space $L^{\Phi,\phi}(\mathbb{R}^n)$.

For the convenience of the reader let us now shortly recapture the main results of this paper.

For the definitions of the spaces appearing in the formulations below, see (3.1) for local (central) Morrey spaces $LM^{p,q}(\mathbb{G})$, (3.2) for generalized local (central) Morrey spaces $LM^{p,\phi}(\mathbb{G})$, and (7.1) for generalized Camponato spaces $\mathcal{LM}^{p,\phi}(\mathbb{G})$, as well as (3.4) for the Hardy-Littlewood maximal operator M, (2.4) for Bessel-Riesz operators $I_{\alpha,\gamma}$, (5.1) for generalized Bessel-Riesz operators $I_{\rho,\gamma}$, and (6.1) for generalized fractional integral operators T_ρ. Both $I_{\rho,\gamma}$ and T_ρ generalise the Riesz transform and the Bessel-Riesz transform in different directions.

Thus, in this paper we show that for a homogeneous group $\mathbb{G}$ of homogeneous dimension Q and any homogeneous quasi-norm $|\cdot|$ we have the following properties:

- If $0 < \alpha < Q$ and $\gamma > 0$, then $K_{\alpha,\gamma} \in L^{p_1}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$, and $\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \sim \left(\sum_{k\in\mathbb{Z}} \frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}}\right)^{\frac{1}{p_1}}$ for any $R > 0$, where $K_{\alpha,\gamma} := \frac{|x|^{\alpha-n}}{(1+|x|)^{\gamma}}$.
- For any $f \in LM^{p,\phi}(\mathbb{G})$ and $1 < p < \infty$, we have
$$\|Mf\|_{LM^{p,\phi}(\mathbb{G})} \leq C_p\|f\|_{LM^{p,\phi}(\mathbb{G})},$$
where generalized local (central) Morrey space $LM^{p,\phi}(\mathbb{G})$ and Hardy-Littlewood maximal operator Mf are defined in (3.2) and (3.4), respectively.
- Let $\gamma > 0$ and $0 < \alpha < Q$. If $\phi(r) \leq Cr^\beta$ for every $r > 0, \beta < -\alpha, 1 < p < \infty$, and $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$, then for all $f \in LM^{p,\phi}(\mathbb{G})$ we have
$$\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},$$
where $q = \frac{\beta p_1' p}{\beta p_1' + Q}$ and $\psi(r) = (\phi(r))^{p/q}$. The Bessel-Riesz operator $I_{\alpha,\gamma}$ on a homogenous group is defined in (2.4).
- Let $\gamma > 0$ and $0 < \alpha < Q$. If $\phi(r) \leq Cr^\beta$ for every $r > 0, \beta < -\alpha$, $\frac{Q}{Q+\gamma-\alpha} < p_2 \leq p_1 < \frac{Q}{Q-\alpha}$ and $p_2 \geq 1$, then for all $f \in LM^{p,\phi}(\mathbb{G})$ we have

$$\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},$$

where $1 < p < \infty, q = \frac{\beta p_1' p}{\beta p_1' + Q}, \psi(r) = (\phi(r))^{p/q}$.

- Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the doubling condition and assume that $\omega(r) \leq Cr^{-\alpha}$ for every $r > 0$, so that $K_{\alpha,\gamma} \in LM^{p_2,\omega}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha} < p_2 < \frac{Q}{Q-\alpha}$ and $p_2 \geq 1$, where $0 < \alpha < Q$ and $\gamma > 0$. If $\phi(r) \leq Cr^{\beta}$ for every $r > 0$, where $\beta < -\alpha < -Q - \beta$, then for all $f \in LM^{p,\phi}(\mathbb{G})$ we have

$$\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},$$

where $1 < p < \infty, q = \frac{\beta p}{\beta+Q-\alpha}$ and $\psi(r) = (\phi(r))^{p/q}$.

- Let $\gamma > 0$ and let ρ and ϕ satisfy the doubling condition (3.3). Let $1 < p < q < \infty$. Let ϕ be surjective and satisfy $\int_r^\infty \frac{\phi(t)^p}{t}dt \leq C_1(\phi(r))^p$, and

$$\phi(r)\int_0^r \frac{\rho(t)}{t^{\gamma-Q+1}}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t^{\gamma-Q+1}}dt \leq C_2(\phi(r))^{p/q},$$

for all $r > 0$. Then we have

$$\|I_{\rho,\gamma}f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})} \leq C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})},$$

where the generalized Bessel-Riesz operator $I_{\rho,\gamma}$ is defined in (5.1). This result is new already in the standard setting of $\mathbb{R}^n$.

- Let ρ and ϕ satisfy the doubling condition (3.3). Let $\gamma > 0$, and assume that ϕ is surjective and satisfies (5.3)-(5.4). Then for $1 < p < p_2 < \infty$ we have

$$\|WI_{\rho,\gamma}f\|_{LM^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},$$

provided that $W \in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$. This result is new even in the Euclidean cases.

- Let ρ and ϕ satisfy the doubling condition (3.3). Let $1 < p < q < \infty$. Let ϕ be surjective and satisfy $\int_r^\infty \frac{\phi(t)^p}{t}dt \leq C_1(\phi(r))^p$, and

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \leq C_2(\phi(r))^{p/q},$$

for all $r > 0$. Then we have

$$\|T_\rho f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})} \leq C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})},$$

where the generalized fractional integral operator T_ρ is defined in (6.1).

- Let ρ and ϕ satisfy the doubling condition (3.3). Let ϕ be surjective and satisfy (6.3)-(6.4). Then for $1 < p < p_2 < \infty$ we have

$$\|WT_\rho f\|_{LM^{p,\phi}(\mathbb{G})} \le C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},$$

provided that $W \in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$.
- Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the doubling condition and assume that $\omega(r) \le Cr^{-\alpha}$ for every $r > 0$, so that $K_{\alpha,\gamma} \in LM^{p_2,\omega}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha} < p_2 < \frac{Q}{Q-\alpha}$ and $p_2 \ge 1$, where $0 < \alpha < Q$, $1 < p < \infty, q = \frac{\beta p}{\beta+Q-\alpha}$ and $\gamma > 0$. If $\phi(r) \le Cr^\beta$ for every $r > 0$, where $\beta < -\alpha < -Q - \beta$, then we have

$$\|WI_{\alpha,\gamma} f\|_{LM^{p,\phi}(\mathbb{G})} \le C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},$$

provided that $W \in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$, where $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{q}$. This result is new already in the Euclidean setting of $\mathbb{R}^n$.
- Let ρ satisfy (6.2), (3.3), (7.3), (7.4), and let ϕ satisfy the doubling condition (3.3) and $\int_1^\infty \frac{\phi(t)}{t}dt < \infty$. If

$$\int_r^\infty \frac{\phi(t)}{t}dt \int_0^r \frac{\rho(t)}{t}dt + r\int_r^\infty \frac{\rho(t)\phi(t)}{t^2}dt \le C_3\psi(r)$$

for all $r > 0$, then we have

$$\|\widetilde{T}_\rho f\|_{\mathcal{LM}^{p,\psi}(\mathbb{G})} \le C_{p,\phi,Q}\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}, \quad 1 < p < \infty,$$

where the generalized local (central) Campanato space $\mathcal{LM}^{p,\psi}(\mathbb{G})$ and operator $\widetilde{T}_\rho$ are defined in (7.1) and (7.2), respectively.

This paper is structured as follows. In Section 2 we briefly recall the concepts of homogeneous groups and fix the notation. The boundedness of the Hardy-Littlewood maximal operator and Bessel-Riesz operators in generalized local (central) Morrey spaces on homogeneous groups is proved in Section 3 and in Section 4, respectively. In Section 5 we prove the boundedness of the generalized Bessel-Riesz operators and Olsen type inequality for these operators in generalized local (central) Morrey spaces on homogeneous groups. The boundedness of the generalized fractional integral operators and Olsen type inequality for these operators in generalized local (central) Morrey spaces on homogeneous groups are proved in Section 6. Finally, in Section 7 we investigate the boundedness of the modified version of the generalized fractional integral operator in Campanato spaces on homogeneous groups.

2. Preliminaries

A connected simply connected Lie group $\mathbb{G}$ is called a *homogeneous group* if its Lie algebra $\mathfrak{g}$ is equipped with a family of dilations:

$$D_\lambda = \mathrm{Exp}(A \ln\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!}(\ln(\lambda)A)^k,$$

where A is a diagonalisable positive linear operator on $\mathfrak{g}$, and each D_λ is a morphism of $\mathfrak{g}$, that is, $\forall X, Y \in \mathfrak{g}$, $\lambda > 0$, $[D_\lambda X, D_\lambda Y] = D_\lambda[X, Y]$.

The exponential mapping $\exp_{\mathbb{G}} : \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism and gives the dilation structure, which is denoted by $D_\lambda x$ or just by λx, on $\mathbb{G}$.

Then we have

$$|D_\lambda(S)| = \lambda^Q |S| \quad \text{and} \quad \int_{\mathbb{G}} f(\lambda x)dx = \lambda^{-Q} \int_{\mathbb{G}} f(x)dx, \tag{2.1}$$

where dx is the Haar measure on $\mathbb{G}$, $|S|$ is the volume of a measurable set $S \subset \mathbb{G}$ and $Q := \mathrm{Tr}\, A$ is the homogeneous dimension of $\mathbb{G}$. Recall that the Haar measure on a homogeneous group $\mathbb{G}$ is the standard Lebesgue measure for $\mathbb{R}^n$ (see e.g. [14, Proposition 1.6.6]).

Let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. We will denote the quasi-ball centred at $x \in \mathbb{G}$ with radius $R > 0$ by

$$B(x, R) := \{y \in \mathbb{G} : |x^{-1}y| < R\}$$

and we will also use the notation

$$B^c(x, R) := \{y \in \mathbb{G} : |x^{-1}y| \geq R\}.$$

The proof of the following important polar decomposition on homogeneous Lie groups was given by Folland and Stein [15], which can be also found in [14, Section 3.1.7]: there is a (unique) positive Borel measure σ on the unit sphere

$$\mathfrak{S} := \{x \in \mathbb{G} : \, |x| = 1\}, \tag{2.2}$$

so that for any $f \in L^1(\mathbb{G})$, one has

$$\int_{\mathbb{G}} f(x)dx = \int_0^\infty \int_{\mathfrak{S}} f(ry)r^{Q-1}d\sigma(y)dr. \tag{2.3}$$

Now, for any $f \in L^p_{loc}(\mathbb{G})$, $p \geq 1$ and $\gamma \geq 0$, $0 < \alpha < Q$, we shall define the Bessel-Riesz operators on homogeneous groups by

$$I_{\alpha,\gamma} f(x) := \int_{\mathbb{G}} K_{\alpha,\gamma}(xy^{-1}) f(y)dy = \int_{\mathbb{G}} \frac{|xy^{-1}|^{\alpha-Q}}{(1 + |xy^{-1}|)^\gamma} f(y)dy, \tag{2.4}$$

where $|\cdot|$ is any homogeneous quasi-norm. Here, $K_{\alpha,\gamma}$ is the Bessel-Riesz kernel. Hereafter, C, C_i, C_p, $C_{p,\phi,Q}$ and $C_{p,q,\phi,Q}$ are positive constants, which are not necessarily the same from line to line.

Let us recall the following result, which will be used in the sequel.

LEMMA 2.1 ([26]). *If $b > a > 0$, then* $\sum_{k\in\mathbb{Z}} \frac{(u^kR)^a}{(1+u^kR)^b} < \infty$, *for every* $u > 1$ *and* $R > 0$.

We now calculate the L^p-norms of the Bessel-Riesz kernel.

THEOREM 2.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm. Let $K_{\alpha,\gamma}(x) = \frac{|x|^{\alpha-Q}}{(1+|x|)^\gamma}$. If $0 < \alpha < Q$ and $\gamma > 0$, then $K_{\alpha,\gamma} \in L^{p_1}(\mathbb{G})$ and*

$$\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \sim \left(\sum_{k\in\mathbb{Z}} \frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}}\right)^{\frac{1}{p_1}},$$

for any $R > 0$ and $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$.

REMARK 2.1. We note that this result was proved in [26, Theorem 3] in the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$ with the standard Euclidean distance $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$.

Proof of Theorem 2.1. Introducing polar coordinates $(r, y) = (|x|, \frac{x}{|x|}) \in (0,\infty) \times \mathfrak{S}$ on $\mathbb{G}$, where $\mathfrak{S}$ is the sphere as in (2.2), and using (2.3) for any $R > 0$, we have

$$\int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx = \int_{\mathbb{G}} \frac{|x|^{(\alpha-Q)p_1}}{(1+|x|)^{\gamma p_1}} dx$$

$$= \int_0^\infty \int_{\mathfrak{S}} \frac{r^{(\alpha-Q)p_1+Q-1}}{(1+r)^{\gamma p_1}} d\sigma(y) dr = |\sigma| \sum_{k\in\mathbb{Z}} \int_{2^kR\le r<2^{k+1}R} \frac{r^{(\alpha-Q)p_1+Q-1}}{(1+r)^{\gamma p_1}} dr,$$

where $|\sigma|$ is the $Q-1$ dimensional surface measure of the unit sphere.

Then it follows that

$$\int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx \le |\sigma| \sum_{k\in\mathbb{Z}} \frac{1}{(1+2^kR)^{\gamma p_1}} \int_{2^kR\le r<2^{k+1}R} r^{(\alpha-Q)p_1+Q-1} dr$$

$$= \frac{|\sigma|(2^{(\alpha-Q)p_1+Q}-1)}{(\alpha-Q)p_1+Q} \sum_{k\in\mathbb{Z}} \frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}}.$$

On the other hand, we obtain

$$\int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx \ge \frac{|\sigma|}{2^{\gamma p_1}} \sum_{k\in\mathbb{Z}} \frac{1}{(1+2^kR)^{\gamma p_1}} \int_{2^kR\le r<2^{k+1}R} r^{(\alpha-Q)p_1+Q-1} dr$$

$$= \frac{|\sigma|(2^{(\alpha-Q)p_1+Q}-1)}{2^{\gamma p_1}((\alpha-Q)p_1+Q)} \sum_{k\in\mathbb{Z}} \frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}}.$$

Therefore, for every $R>0$ we arrive at

$$\int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx \sim \sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}}.$$

For $p_1 \in \left(\frac{Q}{Q+\gamma-\alpha}, \frac{Q}{Q-\alpha}\right)$ using Lemma 2.1 with $u=2, a=(\alpha-Q)p_1+Q, b=\gamma p_1$, we obtain $\sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} < \infty$ which implies $K_{\alpha,\gamma} \in L^{p_1}(\mathbb{G})$. $\square$

The following is well-known on homogeneous groups, see e.g. [14, Proposition 1.5.2].

Proposition 2.1 (Young's inequality). *Let $\mathbb{G}$ be a homogeneous group. Suppose $1 \le p, q, p_1 \le \infty$ and $\frac{1}{q}+1 = \frac{1}{p}+\frac{1}{p_1}$. If $f \in L^p(\mathbb{G})$ and $g \in L^{p_1}(\mathbb{G})$, then*

$$\|g * f\|_{L^q(\mathbb{G})} \le \|f\|_{L^p(\mathbb{G})} \|g\|_{L^{p_1}(\mathbb{G})}.$$

In view of Proposition 2.1 and taking into account the definition of Bessel-Riesz operator (2.4), we immediately get:

Corollary 2.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm. Then for $0<\alpha<Q$ and $\gamma>0$ we have*

$$\|I_{\alpha,\gamma} f\|_{L^q(\mathbb{G})} \le \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{L^p(\mathbb{G})}$$

for every $f \in L^p(\mathbb{G})$, where $1 \le p, q, p_1 \le \infty$, $\frac{1}{q}+1 = \frac{1}{p}+\frac{1}{p_1}$ and $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$.

Corollary 2.1 shows that the $I_{\alpha,\gamma}$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ and

$$\|I_{\alpha,\gamma}\|_{L^p(\mathbb{G}) \to L^q(\mathbb{G})} \le \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}.$$

3. The boundedness of Hardy-Littlewood maximal operator in generalized local (central) Morrey spaces

In this section we define local (central) Morrey and generalized local (central) Morrey spaces on homogeneous groups. Then we prove that the Hardy-Littlewood maximal operator is bounded in these spaces. Note that in the isotropic Abelian case the result was obtained by Nakai [29].

Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let us define the local (central) Morrey spaces $LM^{p,q}(\mathbb{G})$ by

$$LM^{p,q}(\mathbb{G}) := \{f \in L^p_{loc}(\mathbb{G}) : \|f\|_{LM^{p,q}(\mathbb{G})} < \infty\}, \;\; 1 \le p \le q, \tag{3.1}$$

where $\|f\|_{LM^{p,q}(\mathbb{G})} := \sup_{r>0} r^{Q(1/q-1/p)} \left(\int_{B(0,r)} |f(x)|^p dx\right)^{1/p}$. Next, for a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ and $1 \leq p < \infty$, we define the generalized local (central) Morrey space $LM^{p,\phi}(\mathbb{G})$ by

$$LM^{p,\phi}(\mathbb{G}) := \{f \in L^p_{loc}(\mathbb{G}) : \|f\|_{LM^{p,\phi}(\mathbb{G})} < \infty\}, \tag{3.2}$$

where $\|f\|_{LM^{p,\phi}(\mathbb{G})} := \sup_{r>0} \frac{1}{\phi(r)} \left(\frac{1}{r^Q}\int_{B(0,r)} |f(x)|^p dx\right)^{1/p}$. Here we assume that ϕ is nonincreasing and $t^{Q/p}\phi(t)$ is nondecreasing, so that ϕ satisfies the doubling condition, i.e. there exists a constant $C_1 > 0$ such that

$$\frac{1}{2} \leq \frac{r}{s} \leq 2 \Longrightarrow \frac{1}{C_1} \leq \frac{\phi(r)}{\phi(s)} \leq C_1. \tag{3.3}$$

Now, for every $f \in L^p_{loc}(\mathbb{G})$, we define the Hardy-Littlewood maximal operator M by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \tag{3.4}$$

where $|B(x,r)|$ denotes the Haar measure of the (quasi-)ball $B = B(x,r)$.

Using the definition of local Morrey spaces (3.1), one can readily obtain:

Lemma 3.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm. Then*

$$\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})} \leq \|K_{\alpha,\gamma}\|_{LM^{p_1,p_1}(\mathbb{G})} = \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}, \tag{3.5}$$

where $1 \leq p_2 \leq p_1$ and $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$.

We now prove the boundedness of the Hardy-Littlewood maximal operator on generalized local Morrey spaces.

Theorem 3.1. *Let $\mathbb{G}$ be a homogeneous group. For any $f \in LM^{p,\phi}(\mathbb{G})$ and $1 < p < \infty$, we have*

$$\|Mf\|_{LM^{p,\phi}(\mathbb{G})} \leq C_p \|f\|_{LM^{p,\phi}(\mathbb{G})}. \tag{3.6}$$

Remark 3.1. We note that this result was proved on stratified groups (or homogeneous Carnot groups) in [19, Corollary 3.2]. Here, Theorem 3.1 holds on general homogeneous groups.

Proof of Theorem 3.1. By the definition of the norm of the generalized local (central) Morrey space (3.2), we have

$$\|f\|_{LM^{p,\phi}(\mathbb{G})} = \sup_{r>0} \frac{1}{\phi(r)} \left(\frac{1}{r^Q}\int_{B(0,r)} |f(x)|^p dx\right)^{1/p}.$$

This implies that

$$\left(\int_{B(0,r)} |f(x)|^p dx\right)^{1/p} \leq \phi(r) r^{\frac{Q}{p}} \|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{3.7}$$

for any $r > 0$.

On the other hand, using Corollary 2.5 (b) from Folland and Stein [15] we have

$$\begin{aligned}\left(\int_{\mathbb{G}} |M(f\chi_{B(0,r)})(x)|^p dx\right)^{1/p} &\leq C_p \left(\int_{\mathbb{G}} |f(x)\chi_{B(0,r)}|^p dx\right)^{1/p} \\ &= C_p \left(\int_{B(0,r)} |f(x)|^p dx\right)^{1/p},\end{aligned}$$

which implies

$$\left(\int_{B(0,r)} |Mf(x)|^p dx\right)^{1/p} \leq C_p \left(\int_{B(0,r)} |f(x)|^p dx\right)^{1/p}. \tag{3.8}$$

Combining (3.7) and (3.8) we get for all $r > 0$

$$\frac{1}{\phi(r)} \left(\frac{1}{r^Q} \int_{B(0,r)} |Mf(x)|^p dx\right)^{1/p} \leq C_p \|f\|_{LM^{p,\phi}(\mathbb{G})},$$

which implies $\|Mf\|_{LM^{p,\phi}(\mathbb{G})} \leq C_p \|f\|_{LM^{p,\phi}(\mathbb{G})}$, completing the proof. □

4. Inequalities for Bessel-Riesz operators on generalized local Morrey spaces

In this section, we prove the boundedness of the Bessel-Riesz operators on the generalized local (central) Morrey space (3.2). In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$ with the standard Euclidean distance $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ the results of this section were obtained in [27].

THEOREM 4.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm. Let $\gamma > 0$ and $0 < \alpha < Q$. If $\phi(r) \leq Cr^\beta$ for every $r > 0, \beta < -\alpha, 1 < p < \infty$, and $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$, then for all $f \in LM^{p,\phi}(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma} f\|_{LM^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q} \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{4.1}$$

where $q = \frac{\beta p_1' p}{\beta p_1' + Q}$ and $\psi(r) = (\phi(r))^{p/q}$.

Proof of Theorem 4.1. For every $f \in LM^{p,\phi}(\mathbb{G})$, we write $I_{\alpha,\gamma} f(x)$ in the form $I_{\alpha,\gamma} f(x) := I_1(x) + I_2(x)$, where $I_1(x) := \int_{B(x,R)} \frac{|xy^{-1}|^{\alpha-Q} f(y)}{(1+|xy^{-1}|)^\gamma} dy$ and $I_2(x) := \int_{B^c(x,R)} \frac{|xy^{-1}|^{\alpha-Q} f(y)}{(1+|xy^{-1}|)^\gamma} dy$, for some $R > 0$.

By using dyadic decomposition for I_1, we obtain

$$
\begin{aligned}
|I_1(x)| &\le \sum_{k=-\infty}^{-1} \int_{2^kR\le|xy^{-1}|<2^{k+1}R} \frac{|xy^{-1}|^{\alpha-Q}|f(y)|}{(1+|xy^{-1}|)^\gamma} dy \\
&\le \sum_{k=-\infty}^{-1} \frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^\gamma} \int_{2^kR\le|xy^{-1}|<2^{k+1}R} |f(y)|dy \\
&\le CMf(x) \sum_{k=-\infty}^{-1} \frac{(2^kR)^{\alpha-Q+Q/p_1}(2^kR)^{Q/p_1'}}{(1+2^kR)^\gamma}.
\end{aligned}
$$

From this using Hölder's inequality for $\frac{1}{p_1}+\frac{1}{p_1'}=1$, we get

$$
|I_1(x)| \le CMf(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}} \right)^{1/p_1} \left(\sum_{k=-\infty}^{-1} (2^kR)^Q \right)^{1/p_1'}.
$$

Since

$$
\begin{aligned}
\left(\sum_{k=-\infty}^{-1} \frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}} \right)^{1/p_1} &\le \left(\sum_{k\in\mathbb{Z}} \frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}} \right)^{1/p_1} \\
&\sim \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}, \quad (4.2)
\end{aligned}
$$

we arrive at

$$
|I_1(x)| \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} Mf(x) R^{Q/p_1'}. \tag{4.3}
$$

For the second term I_2, by using Hölder's inequality for $\frac{1}{p}+\frac{1}{p'}=1$ we obtain that

$$
\begin{aligned}
|I_2(x)| \le \sum_{k=0}^{\infty} \frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^\gamma} \int_{2^kR\le|xy^{-1}|<2^{k+1}R} |f(y)|dy &\le \sum_{k=0}^{\infty} \frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^\gamma} \\
\times \left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R} dy \right)^{1/p'} \left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R} |f(y)|^p dy \right)^{1/p}&,
\end{aligned}
$$

that is,

$$
\begin{aligned}
|I_2(x)| &\le \sum_{k=0}^{\infty} \frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^\gamma} \left(\int_{2^kR}^{2^{k+1}R} \int_{\mathfrak{S}} r^{Q-1} d\sigma(y) dr \right)^{1/p'} \\
&\quad \times \left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R} |f(y)|^p dy \right)^{1/p} \\
&\le C \sum_{k=0}^{\infty} \frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^\gamma} (2^kR)^{Q/p'} \left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R} |f(y)|^p dy \right)^{1/p}.
\end{aligned}
$$

Taking into account $\phi(r) \le Cr^{\beta}$, one obtains from above that

$$|I_2(x)| \le C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q+Q/p_1}}{(1+2^k R)^{\gamma}} \phi(2^k R)(2^k R)^{Q/p_1'}$$
$$\le C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q+Q/p_1}}{(1+2^k R)^{\gamma}} (2^k R)^{\beta+Q/p_1'}.$$

Applying Hölder's inequality again, we get

$$|I_2(x)| \le C\|f\|_{LM^{p,\phi}(\mathbb{G})} \left(\sum_{k=0}^{\infty} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{1/p_1} \left(\sum_{k=0}^{\infty} (2^k R)^{\beta p_1'+Q} \right)^{1/p_1'}.$$

From the conditions $p_1 < \frac{Q}{Q-\alpha}$ and $\beta < -\alpha$, we have $\beta p_1' + Q < 0$. By Theorem 2.1, we also have

$$\left(\sum_{k=0}^{\infty} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{1/p_1} \le \left(\sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{1/p_1} \sim \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}.$$

Using these, we arrive at

$$|I_2(x)| \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})} R^{Q/p_1'+\beta}. \tag{4.4}$$

Summing up the estimates (4.3) and (4.4), we obtain

$$|I_{\alpha,\gamma}f(x)| \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \left(Mf(x) R^{Q/p_1'} + \|f\|_{LM^{p,\phi}(\mathbb{G})} R^{Q/p_1'+\beta} \right).$$

Assuming that f is not identically 0 and that Mf is finite everywhere, we can choose $R > 0$ such that $R^{\beta} = \frac{Mf(x)}{\|f\|_{LM^{p,\phi}(\mathbb{G})}}$, that is,

$$|I_{\alpha,\gamma}f(x)| \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})}^{-\frac{Q}{\beta p_1'}} (Mf(x))^{1+\frac{Q}{\beta p_1'}},$$

for every $x \in \mathbb{G}$. Setting $q = \frac{\beta p_1' p}{\beta p_1' + Q}$, for any $r > 0$ we get

$$\left(\int_{|x|<r} |I_{\alpha,\gamma}f(x)|^q dx \right)^{\frac{1}{q}} \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q}$$
$$\times \left(\int_{|x|<r} |Mf(x)|^p dx \right)^{1/q}.$$

Then we divide both sides by $(\phi(r))^{p/q} r^{Q/q}$ to get

$$\frac{\left(\int_{|x|<r} |I_{\alpha,\gamma} f(x)|^q dx\right)^{\frac{1}{q}}}{\psi(r) r^{Q/q}} \leq C \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q} \times \frac{\left(\int_{|x|<r} |Mf(x)|^p dx\right)^{1/q}}{(\phi(r))^{p/q} r^{Q/q}},$$

where $\psi(r) = (\phi(r))^{p/q}$. Now by taking the supremum over $r > 0$, we obtain that

$$\|I_{\alpha,\gamma} f\|_{LM^{q,\psi}(\mathbb{G})} \leq C \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q} \|Mf\|_{LM^{p,\phi}(\mathbb{G})}^{p/q},$$

which gives (4.1), after applying estimate (3.6). □

Lemma 3.1 gives the property that the Bessel-Riesz kernel belongs to local Morrey spaces, which will be used in the following theorem.

THEOREM 4.2. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm. Let $\gamma > 0$ and $0 < \alpha < Q$. If $\phi(r) \leq Cr^\beta$ for every $r > 0, \beta < -\alpha$, $\frac{Q}{Q+\gamma-\alpha} < p_2 \leq p_1 < \frac{Q}{Q-\alpha}$ and $p_2 \geq 1$, then for all $f \in LM^{p,\phi}(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma} f\|_{LM^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q} \|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{4.5}$$

where $1 < p < \infty, q = \frac{\beta p_1' p}{\beta p_1' + Q}$ and $\psi(r) = (\phi(r))^{p/q}$.

Proof of Theorem 4.2. Similarly to the proof of Theorem 4.1, we write $I_{\alpha,\gamma} f(x) := I_1(x) + I_2(x)$, where

$$I_1(x) := \int_{B(x,R)} \frac{|xy^{-1}|^{\alpha-Q}}{(1+|xy^{-1}|)^\gamma} f(y) dy$$

and

$$I_2(x) := \int_{B^c(x,R)} \frac{|xy^{-1}|^{\alpha-Q}}{(1+|xy^{-1}|)^\gamma} f(y) dy$$

for $R > 0$. As before, we estimate the first term I_1 by using the dyadic decomposition:

$$\begin{aligned}|I_1(x)| &\le \sum_{k=-\infty}^{-1} \int_{2^kR\le|xy^{-1}|<2^{k+1}R} \frac{|xy^{-1}|^{\alpha-Q}|f(y)|}{(1+|xy^{-1}|)^{\gamma}}dy \\ &\le \sum_{k=-\infty}^{-1} \frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^{\gamma}} \int_{2^kR\le|xy^{-1}|<2^{k+1}R} |f(y)|dy \\ &\le CMf(x) \sum_{k=-\infty}^{-1} \frac{(2^kR)^{\alpha-Q+Q/p_2}(2^kR)^{Q/p_2'}}{(1+2^kR)^{\gamma}},\end{aligned}$$

where $1 \le p_2 \le p_1$. From this using Hölder's inequality for $\frac{1}{p_2}+\frac{1}{p_2'}=1$, we get

$$|I_1(x)| \le CMf(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^kR)^{(\alpha-Q)p_2+Q}}{(1+2^kR)^{\gamma p_2}} \right)^{1/p_2} \left(\sum_{k=-\infty}^{-1} (2^kR)^{Q} \right)^{1/p_2'}.$$

By virtue of (4.2), we have

$$\begin{aligned}|I_1(x)| \le C_2Mf(x) &\left(\int_{0<|x|<R} K_{\alpha,\gamma}^{p_2}(x)dx \right)^{\frac{1}{p_2}} R^{Q/p_2'} \\ &\le C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})} Mf(x) R^{Q/p_1'}. \qquad (4.6)\end{aligned}$$

Now for I_2 by using Hölder's inequality for $\frac{1}{p}+\frac{1}{p'}=1$, we have

$$|I_2(x)| \le \sum_{k=0}^{\infty} \frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^{\gamma}} (2^kR)^{Q/p'} \left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R} |f(y)|^p dy \right)^{1/p},$$

that is,

$$\begin{aligned}|I_2(x)| &\le C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^kR)^{\alpha}\phi(2^kR)}{(1+2^kR)^{\gamma}} \frac{\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R} dy\right)^{1/p_2}}{(2^kR)^{Q/p_2}} \\ &\le C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \phi(2^kR)(2^kR)^{Q/p_1'} \frac{\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R} K_{\alpha,\gamma}^{p_2}(xy^{-1})dy\right)^{1/p_2}}{(2^kR)^{Q/p_2-Q/p_1}},\end{aligned}$$

where we have used the following inequality

$$\begin{aligned}&\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R} K_{\alpha,\gamma}^{p_2}(xy^{-1})dy \right)^{1/p_2} \\ &\sim \frac{(2^kR)^{(\alpha-Q)+Q/p_2}}{(1+2^kR)^{\gamma}} \ge C\frac{(2^kR)^{(\alpha-Q)}}{(1+2^kR)^{\gamma}} \left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R} dy \right)^{1/p_2}. \qquad (4.7)\end{aligned}$$

Since we have $\phi(r) \leq Cr^{\beta}$ and

$$\frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} K_{\alpha,\gamma}^{p_2}(xy^{-1})dy\right)^{1/p_2}}{(2^k R)^{Q/p_2 - Q/p_1}} \lesssim \|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}$$

for every $k = 0, 1, 2, ...$, we get

$$|I_2(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} (2^k R)^{\beta + Q/p_1^{'}}.$$

Taking into account $\beta + Q/p_1^{'} < 0$, we have

$$|I_2(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})} R^{\beta + Q/p_1^{'}}. \tag{4.8}$$

Summing up the estimates (4.6) and (4.8), we obtain

$$|I_{\alpha,\gamma}f(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}(Mf(x)R^{Q/p_1^{'}} + \|f\|_{LM^{p,\phi}(\mathbb{G})} R^{\beta + Q/p_1^{'}}).$$

Assuming that f is not identically 0 and that Mf is finite everywhere, we can choose $R > 0$ such that $R^{\beta} = \frac{Mf(x)}{\|f\|_{LM^{p,\phi}(\mathbb{G})}}$, which yields

$$|I_{\alpha,\gamma}f(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{-\frac{Q}{\beta p_1^{'}}} (Mf(x))^{1+\frac{Q}{\beta p_1^{'}}}.$$

Now by putting $q = \frac{\beta p_1^{'} p}{\beta p_1^{'} + Q}$, for any $r > 0$ we obtain

$$\left(\int_{|x|<r} |I_{\alpha,\gamma}f(x)|^q dx\right)^{\frac{1}{q}} \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q} \times \left(\int_{|x|<r} |Mf(x)|^p dx\right)^{1/q}.$$

Then we divide both sides by $(\phi(r))^{p/q} r^{Q/q}$ to get

$$\frac{\left(\int_{|x|<r} |I_{\alpha,\gamma}f(x)|^q dx\right)^{\frac{1}{q}}}{\psi(r) r^{Q/q}} \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q} \times \frac{\left(\int_{|x|<r} |Mf(x)|^p dx\right)^{1/q}}{(\phi(r))^{p/q} r^{Q/q}},$$

where $\psi(r) = (\phi(r))^{p/q}$. Taking the supremum over $r > 0$ and then using (3.6), we obtain the following desired result

$$\begin{aligned}\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} &\leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q}\|Mf\|_{LM^{p,\phi}(\mathbb{G})}^{p/q} \\ &\leq C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},\end{aligned}$$

completing the proof. □

By Lemma 3.1, we note that Theorem 4.2 implies Theorem 4.1:

$$\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}$$
$$\leq C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}.$$

In order to improve our results, we present the following lemma, which states that the kernel $K_{\alpha,\gamma}$ belongs to the generalized local Morrey space $LM^{p_2,\omega}(\mathbb{G})$ for some $p_2 \geq 1$ and some function ω.

LEMMA 4.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $\gamma > 0$, $p_2 \geq 1$ and $Q - \frac{Q}{p_2} < \alpha < Q$. If $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(r) \geq Cr^{\alpha-Q}$ for every $r > 0$, then $K_{\alpha,\gamma} \in LM^{p_2,\omega}(\mathbb{G})$.*

Proof of Lemma 4.1. Here, it is sufficient to evaluate the following integral around zero

$$\int_{|x|\leq R} K^{p_2}_{\alpha,\gamma}(x)dx = \int_{|x|\leq R} \frac{|x|^{(\alpha-Q)p_2}}{(1+|x|)^{\gamma p_2}}dx$$
$$\leq |\sigma| \int_{0<r\leq R} r^{(\alpha-Q)p_2+Q-1}dr \leq C\omega^{p_2}(R)R^Q.$$

By dividing both sides of this inequality by $\omega^{p_2}(R)R^Q$ and taking p_2^{th}-root, we obtain

$$\frac{\left(\int_{|x|\leq R} K^{p_2}_{\alpha,\gamma}(x)dx\right)^{1/p_2}}{\omega(R)R^{Q/p_2}} \leq C^{1/p_2}.$$

Then, we take the supremum over $R > 0$ to get

$$\sup_{R>0} \frac{\left(\int_{|x|\leq R} K^{p_2}_{\alpha,\gamma}(x)dx\right)^{1/p_2}}{\omega(R)R^{Q/p_2}} < \infty,$$

which implies $K_{\alpha,\gamma} \in LM^{p_2,\omega}(\mathbb{G})$. □

THEOREM 4.3. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the doubling condition and assume that $\omega(r) \leq Cr^{-\alpha}$ for every $r > 0$, so that $K_{\alpha,\gamma} \in LM^{p_2,\omega}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha} < p_2 < \frac{Q}{Q-\alpha}$ and $p_2 \geq 1$, where $0 < \alpha < Q$ and $\gamma > 0$. If $\phi(r) \leq Cr^{\beta}$ for every $r > 0$, where $\beta < -\alpha < -Q - \beta$, then for all $f \in LM^{p,\phi}(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{4.9}$$

where $1 < p < \infty, q = \frac{\beta p}{\beta+Q-\alpha}$ and $\psi(r) = (\phi(r))^{p/q}$.

Proof of Theorem 4.3. As in the proof of Theorem 4.1, we write

$$I_{\alpha,\gamma}f(x) := I_1(x) + I_2(x),$$

where $I_1(x) := \int_{B(x,R)} \frac{|xy^{-1}|^{\alpha-Q} f(y)}{(1+|xy^{-1}|)^{\gamma}} dy$ and $I_2(x) := \int_{B^c(x,R)} \frac{|xy^{-1}|^{\alpha-Q} f(y)}{(1+|xy^{-1}|)^{\gamma}} dy$, $R > 0$.

First, we estimate I_1 by using the dyadic decomposition

$$\begin{aligned}|I_1(x)| &\leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} \frac{|xy^{-1}|^{\alpha-Q}|f(y)|}{(1+|xy^{-1}|)^{\gamma}} dy \\ &\leq \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^{\gamma}} \int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} |f(y)| dy \\ &\leq CMf(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-Q+Q/p_2} (2^k R)^{Q/p_2'}}{(1+2^k R)^{\gamma}}.\end{aligned}$$

From this using Hölder's inequality for $\frac{1}{p_2} + \frac{1}{p_2'} = 1$, we get

$$|I_1(x)| \leq CMf(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-Q)p_2+Q}}{(1+2^k R)^{\gamma p_2}} \right)^{1/p_2} \left(\sum_{k=-\infty}^{-1} (2^k R)^Q \right)^{1/p_2'}.$$

By (4.2) we have

$$\begin{aligned}|I_1(x)| &\leq CMf(x) \left(\int_{0<|x|<R} K_{\alpha,\gamma}^{p_2}(x) dx \right)^{\frac{1}{p_2}} R^{Q/p_2'} \\ &\leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})} Mf(x) \omega(R) R^Q,\end{aligned}$$

and using $\omega(r) \leq Cr^{-\alpha}$, we arrive at

$$|I_1(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})} Mf(x) R^{Q-\alpha}. \tag{4.10}$$

Now let us estimate the second term I_2:

$$\begin{aligned}|I_2(x)| &\leq \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^{\gamma}} \int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} |f(y)| dy \\ &\leq C \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^{\gamma}} (2^k R)^{Q/p'} \left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} |f(y)|^p dy \right)^{1/p} \\ &\leq C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha} \phi(2^k R)}{(1+2^k R)^{\gamma}} \frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} dy \right)^{1/p_2}}{(2^k R)^{Q/p_2}},\end{aligned}$$

where we have used that $\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} dy \right)^{1/p_2} \sim (2^k R)^{Q/p_2}$. Using (4.7) we obtain

$$|I_2(x)| \leq C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha}\phi(2^k R)}{(2^k R)^{\alpha-Q}} \times \frac{\left(\int_{2^k R\leq |xy^{-1}|<2^{k+1}R} K_{\alpha,\gamma}^{p_2}(xy^{-1})dy\right)^{1/p_2}}{(2^k R)^{Q/p_2}}.$$

Taking into account that $\phi(r) \leq Cr^{\beta}$ and $\omega(r) \leq Cr^{-\alpha}$ for every $r > 0$, we have

$$|I_2(x)| \leq C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} (2^k R)^{Q-\alpha+\beta} \frac{\left(\int_{2^k R\leq |xy^{-1}|<2^{k+1}R} K_{\alpha,\gamma}^{p_2}(xy^{-1})dy\right)^{1/p_2}}{\omega(2^k R)(2^k R)^{Q/p_2}}.$$

Since we have

$$\frac{\left(\int_{2^k R\leq |xy^{-1}|<2^{k+1}R} K_{\alpha,\gamma}^{p_2}(xy^{-1})dy\right)^{1/p_2}}{\omega(2^k R)(2^k R)^{Q/p_2}} \lesssim \|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}$$

for every $k = 0, 1, 2, \ldots$, it follows that

$$|I_2(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} (2^k R)^{Q-\alpha+\beta},$$

and since $Q - \alpha + \beta < 0$, it implies that

$$|I_2(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})} R^{Q-\alpha+\beta}. \tag{4.11}$$

Summing up the estimates (4.10) and (4.11), we have

$$|I_{\alpha,\gamma}f(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}(Mf(x)R^{Q-\alpha} + \|f\|_{LM^{p,\phi}(\mathbb{G})}R^{Q-\alpha+\beta}).$$

Assuming that f is not identically 0 and that Mf is finite everywhere, we can choose $R > 0$ such that $R^{\beta} = \frac{Mf(x)}{\|f\|_{LM^{p,\phi}(\mathbb{G})}}$, that is

$$|I_{\alpha,\gamma}f(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{(\alpha-Q)/\beta}(Mf(x))^{1+(Q-\alpha)/\beta}.$$

Now by putting $q = \frac{\beta p}{\beta+Q-\alpha}$, for any $r > 0$ we get

$$\left(\int_{|x|<r} |I_{\alpha,\gamma}f(x)|^q dx\right)^{\frac{1}{q}} \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q} \times \left(\int_{|x|<r} |Mf(x)|^p dx\right)^{1/q}.$$

Then we divide both sides by $(\phi(r))^{p/q}r^{Q/q}$ to get

$$\frac{\left(\int_{|x|<r} |I_{\alpha,\gamma}f(x)|^q dx\right)^{\frac{1}{q}}}{\psi(r)r^{Q/q}} \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|^{1-p/q}_{LM^{p,\phi}(\mathbb{G})} \times \frac{\left(\int_{|x|<r} |Mf(x)|^p dx\right)^{1/q}}{(\phi(r))^{p/q}r^{Q/q}},$$

where $\psi(r) = (\phi(r))^{p/q}$. Finally, taking the supremum over $r > 0$ and using (3.6), we obtain the desired result

$$\begin{aligned}\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} &\leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|^{1-p/q}_{LM^{p,\phi}(\mathbb{G})}\|Mf\|^{p/q}_{LM^{p,\phi}(\mathbb{G})} \\ &\leq C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},\end{aligned}$$

completing the proof. □

REMARK 4.1. We can make the following comparison between the obtained estimates, similarly to the Euclidean case [27, Section 3], namely, that also in the case of general homogeneous groups, Theorem 4.3 gives the best estimate among the three. Indeed, if we take $\omega(R) := (1 + R^{Q/q_1})R^{-Q/p_1}$ for some $q_1 > p_1$, then $\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})} \leq \|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}$. By Theorem 4.3 and Lemma 3.1 we obtain

$$\begin{aligned}\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} &\leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})} \\ &\leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})} \\ &\leq C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}.\end{aligned}$$

5. Inequalities for generalized Bessel-Riesz operator in generalized local Morrey spaces

In this section, we prove the boundedness of the generalized Bessel-Riesz operator $I_{\widetilde{\rho},\gamma}$ and establish Olsen type inequality for this operator in generalized local Morrey spaces on homogeneous groups.

We define the generalized Bessel-Riesz operator $I_{\widetilde{\rho},\gamma}$ by

$$I_{\widetilde{\rho},\gamma}f(x) := \int_{\mathbb{G}} \frac{\widetilde{\rho}(|xy^{-1}|)}{(1+|xy^{-1}|)^\gamma} f(y)dy, \tag{5.1}$$

where $\gamma \geq 0$, $\widetilde{\rho} : \mathbb{R}^+ \to \mathbb{R}^+$, $\widetilde{\rho}$ satisfies the doubling condition (3.3) and the condition

$$\int_0^1 \frac{\widetilde{\rho}(t)}{t^{\gamma-Q+1}}dt < \infty. \tag{5.2}$$

For $\widetilde{\rho}(t) = t^{\alpha-Q}, \gamma < \alpha < Q$, we have the Bessel-Riesz kernel

$$I_{\widetilde{\rho},\gamma}=I_{\alpha,\gamma}=\frac{|xy^{-1}|^{\alpha-Q}}{(1+|xy^{-1}|)^{\gamma}}.$$

THEOREM 5.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm and let $\gamma>0$. Let $\widetilde{\rho}$ and ϕ satisfy the doubling condition (3.3). Let ϕ be surjective and for some $1<p<q<\infty$ satisfy*

$$\int_r^\infty \frac{(\phi(t))^p}{t}dt\le C_1(\phi(r))^p, \tag{5.3}$$

and

$$\phi(r)\int_0^r \frac{\widetilde{\rho}(t)}{t^{\gamma-Q+1}}dt+\int_r^\infty \frac{\widetilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}}dt\le C_2(\phi(r))^{p/q}, \tag{5.4}$$

for all $r>0$. Then we have

$$\|I_{\widetilde{\rho},\gamma}f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})}\le C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})}. \tag{5.5}$$

Proof of Theorem 5.1. We write $I_{\widetilde{\rho},\gamma}f(x)=I_{1,\widetilde{\rho}}(x)+I_{2,\widetilde{\rho}}(x)$, where $I_{1,\widetilde{\rho}}(x):=\int_{B(x,R)}\frac{\widetilde{\rho}(|xy^{-1}|)f(y)}{(1+|xy^{-1}|)^{\gamma}}dy$ and $I_{2,\widetilde{\rho}}(x):=\int_{B^c(x,R)}\frac{\widetilde{\rho}(|xy^{-1}|)f(y)}{(1+|xy^{-1}|)^{\gamma}}dy$ for every $R>0$. For $I_{1,\widetilde{\rho}}(x)$, we have

$$\begin{aligned}|I_{1,\widetilde{\rho}}(x)|&\le\int_{|xy^{-1}|<R}\frac{\widetilde{\rho}(|xy^{-1}|)}{(1+|xy^{-1}|)^{\gamma}}|f(y)|dy\le\int_{|xy^{-1}|<R}\frac{\widetilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^{\gamma}}|f(y)|dy\\&=\sum_{k=-\infty}^{-1}\int_{2^kR\le|xy^{-1}|<2^{k+1}R}\frac{\widetilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^{\gamma}}|f(y)|dy.\end{aligned}$$

By virtue of (3.3), we get

$$\begin{aligned}|I_{1,\widetilde{\rho}}(x)|&\le C\sum_{k=-\infty}^{-1}\frac{\widetilde{\rho}(2^kR)}{(2^kR)^{\gamma}}\int_{|xy^{-1}|<2^{k+1}R}|f(y)|dy\\&\le CMf(x)\sum_{k=-\infty}^{-1}\frac{\widetilde{\rho}(2^kR)}{(2^kR)^{\gamma-Q}}\\&\le CMf(x)\sum_{k=-\infty}^{-1}\int_{2^kR}^{2^{k+1}R}\frac{\widetilde{\rho}(t)}{t^{\gamma-Q+1}}dt\\&=CMf(x)\int_0^R\frac{\widetilde{\rho}(t)}{t^{\gamma-Q+1}}dt,\end{aligned}$$

where we have used the fact that

$$\int_{2^kR}^{2^{k+1}R}\frac{\widetilde{\rho}(t)}{t^{\gamma-Q+1}}dt\ge C\frac{\widetilde{\rho}(2^kR)}{(2^kR)^{\gamma-Q+1}}2^kR\ge C\frac{\widetilde{\rho}(2^kR)}{(2^kR)^{\gamma-Q}}. \tag{5.6}$$

Now, using (5.4), we obtain

$$|I_{1,\widetilde{\rho}}(x)| \le CMf(x)(\phi(R))^{(p-q)/q}. \tag{5.7}$$

For $I_{2,\widetilde{\rho}}(x)$, applying (3.3) we have

$$\begin{aligned}
|I_{2,\widetilde{\rho}}(x)| &\le \int_{|xy^{-1}|\ge R} \frac{\widetilde{\rho}(|xy^{-1}|)}{(1+|xy^{-1}|)^{\gamma}}|f(y)|dy \le \int_{|xy^{-1}|\ge R} \frac{\widetilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^{\gamma}}|f(y)|dy \\
&= \sum_{k=0}^{\infty} \int_{2^kR\le|xy^{-1}|<2^{k+1}R} \frac{\widetilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^{\gamma}}|f(y)|dy \\
&\le C\sum_{k=0}^{\infty} \frac{\widetilde{\rho}(2^kR)}{(2^kR)^{\gamma}} \int_{|xy^{-1}|<2^{k+1}R} |f(y)|dy.
\end{aligned}$$

From this using the Hölder inequality, we obtain

$$\begin{aligned}
|I_{2,\widetilde{\rho}}(x)| &\le C\sum_{k=0}^{\infty} \frac{\widetilde{\rho}(2^kR)}{(2^kR)^{\gamma}} \left(\int_{|xy^{-1}|<2^{k+1}R} dy\right)^{1-\frac{1}{p}} \left(\int_{|xy^{-1}|<2^{k+1}R} |f(y)|dy\right)^{\frac{1}{p}} \\
&\le C\sum_{k=0}^{\infty} \frac{\widetilde{\rho}(2^kR)}{(2^kR)^{\gamma-Q+\frac{Q}{p}}} \left(\int_{|xy^{-1}|<2^{k+1}R} |f(y)|dy\right)^{\frac{1}{p}}.
\end{aligned}$$

Using the definition (3.2), one gets

$$\begin{aligned}
|I_{2,\widetilde{\rho}}(x)| &\le C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{\widetilde{\rho}(2^{k+1}R)\phi(2^{k+1}R)}{(2^kR)^{\gamma-Q}} \\
&\le C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \int_{2^kR}^{2^{k+1}R} \frac{\widetilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}}dt \\
&= C\|f\|_{LM^{p,\phi}(\mathbb{G})} \int_{R}^{\infty} \frac{\widetilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}}dt,
\end{aligned}$$

where we have used the fact that

$$\int_{2^kR}^{2^{k+1}R} \frac{\widetilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}}dt \ge C\frac{\widetilde{\rho}(2^{k+1}R)\phi(2^{k+1}R)}{(2^{k+1}R)^{\gamma-Q+1}}2^kR \ge C\frac{\widetilde{\rho}(2^{k+1}R)\phi(2^{k+1}R)}{(2^kR)^{\gamma-Q}}.$$

Now, using (5.4), we obtain

$$|I_{2,\widetilde{\rho}}(x)| \le C\|f\|_{LM^{p,\phi}(\mathbb{G})}(\phi(R))^{p/q}. \tag{5.8}$$

Summing the two estimates (5.7) and (5.8), we arrive at

$$|I_{\widetilde{\rho},\gamma}f(x)| \le C(Mf(x)(\phi(R))^{(p-q)/q} + \|f\|_{LM^{p,\phi}(\mathbb{G})}(\phi(R))^{p/q}).$$

Assuming that f is not identically 0 and that Mf is finite everywhere and then using the fact that ϕ is surjective, we can choose $R > 0$ such that $\phi(R) = Mf(x)\,\|f\|^{-1}_{LM^{p,\phi}(\mathbb{G})}$. Thus, for every $x \in \mathbb{G}$, we have

$$|I_{\widetilde{\rho},\gamma}f(x)| \leq C(Mf(x))^{\frac{p}{q}}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}}.$$

It follows that

$$\left(\int_{B(0,r)}|I_{\widetilde{\rho},\gamma}f(x)|^q\right)^{1/q} \leq C\left(\int_{B(0,r)}|Mf(x)|^p\right)^{1/q}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}},$$

then we divide both sides by $(\phi(r))^{p/q}r^{Q/q}$ to get

$$\frac{1}{(\phi(r))^{p/q}}\left(\frac{1}{r^Q}\int_{B(0,r)}|I_{\widetilde{\rho},\gamma}f(x)|^q\right)^{1/q}$$
$$\leq C\frac{1}{(\phi(r))^{p/q}}\left(\frac{1}{r^Q}\int_{B(0,r)}|Mf(x)|^p\right)^{1/q}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}}.$$

Taking the supremum over $r>0$ and using the boundedness of the maximal operator M on $LM^{p,\phi}(\mathbb{G})$ from (3.6), we obtain

$$\|I_{\widetilde{\rho},\gamma}f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})} \leq C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})}.$$

This completes the proof. □

Now let show the Olsen type inequalities for the generalized Bessel-Riesz operator $I_{\rho,\gamma}$.

THEOREM 5.2. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm and let $\gamma>0$. Let $\widetilde{\rho}$ and ϕ satisfy the doubling condition (3.3). Let ϕ be surjective and satisfy (5.3)-(5.4). Then for $1<p<p_2<\infty$ we have*

$$\|WI_{\widetilde{\rho},\gamma}f\|_{LM^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{5.9}$$

provided that $W \in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$.

Proof of Theorem 5.2. By using Hölder's inequality, we have

$$\frac{1}{r^Q}\int_{B(0,r)}|W(x)I_{\widetilde{\rho},\gamma}f(x)|^p dx \leq \left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{p/p_2}$$
$$\times\left(\frac{1}{r^Q}\int_{B(0,r)}|I_{\widetilde{\rho},\gamma}f(x)|^{\frac{pp_2}{p_2-p}}dx\right)^{\frac{p_2-p}{p_2}}.$$

Now let us take the p-th roots and then divide both sides by $\phi(r)$ to obtain

$$
\begin{aligned}
&\frac{1}{\phi(r)}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)I_{\widetilde{\rho},\gamma}f(x)|^p dx\right)^{1/p}\\
&\qquad\le \frac{1}{(\phi(r))^{p/p_2}}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{1/p_2}\\
&\qquad\qquad\times\frac{1}{(\phi(r))^{\frac{p_2-p}{p_2}}}\left(\frac{1}{r^Q}\int_{B(0,r)}|I_{\widetilde{\rho},\gamma}f(x)|^{\frac{pp_2}{p_2-p}}dx\right)^{\frac{p_2-p}{pp_2}}.
\end{aligned}
$$

By taking the supremum over $r>0$ and using the inequality (5.5), we get

$$\|WI_{\widetilde{\rho},\gamma}f\|_{LM^{p,\phi}(\mathbb{G})}\le C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|I_{\widetilde{\rho},\gamma}f\|_{L^{\frac{pp_2}{p_2-p},\phi^{\frac{p_2-p}{p_2}}}(\mathbb{G})}.$$

Taking into account that $1<p<\frac{pp_2}{p_2-p}<\infty$ and putting $q=\frac{pp_2}{p_2-p}$ in (5.5), we obtain (5.9). □

6. Generalized fractional integral operators in generalized local Morrey spaces

In this section, we prove the boundedness of the generalized fractional integral operators and establish Olsen type inequality in generalized local Morrey spaces on homogeneous groups.

We define the generalized fractional integral operator T_ρ by

$$T_\rho f(x):=\int_{\mathbb{G}}\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}f(y)dy, \tag{6.1}$$

where $\rho:\mathbb{R}^+\to\mathbb{R}^+$ satisfies the doubling condition (3.3) and the condition

$$\int_0^1\frac{\rho(t)}{t}dt<\infty. \tag{6.2}$$

As in the Abelian case, for $\rho(t)=t^\alpha$, $0<\alpha<Q$, we have the Riesz transform

$$T_\rho f(x)=I_\alpha f(x)=\int_{\mathbb{G}}\frac{1}{|xy^{-1}|^{Q-\alpha}}f(y)dy.$$

Theorem 6.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm. Let ρ and ϕ satisfy the doubling condition (3.3). Let ϕ be also surjective and satisfy, for some $1<p<q<\infty$, the inequalities*

$$\int_r^\infty\frac{\phi(t)^p}{t}dt\le C_1(\phi(r))^p, \tag{6.3}$$

and

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C_2(\phi(r))^{p/q}, \tag{6.4}$$

for all $r > 0$. Then we have

$$\|T_\rho f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})} \le C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})}. \tag{6.5}$$

P r o o f of T h e o r e m 6.1. For every $R > 0$, let us write $T_\rho f(x)$ in the form

$$T_\rho f(x) = T_1(x) + T_2(x),$$

where $T_1(x) := \int_{B(x,R)} \frac{\rho(|xy^{-1}|)}{(|xy^{-1}|)^Q} f(y)dy$ and $T_2(x) := \int_{B^c(x,R)} \frac{\rho(|xy^{-1}|)}{(|xy^{-1}|)^Q} f(y)dy$. For $T_1(x)$, we have

$$\begin{aligned}|T_1(x)| &\le \int_{|xy^{-1}|<R} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}|f(y)|dy \\ &= \sum_{k=-\infty}^{-1} \int_{2^kR\le|xy^{-1}|<2^{k+1}R} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}|f(y)|dy.\end{aligned}$$

By view of (3.3), we get

$$\begin{aligned}|T_1(x)| &\le C\sum_{k=-\infty}^{-1} \frac{\rho(2^kR)}{(2^kR)^Q}\int_{|xy^{-1}|<2^{k+1}R}|f(y)|dy \le CMf(x)\sum_{k=-\infty}^{-1}\rho(2^kR) \\ &\le CMf(x)\sum_{k=-\infty}^{-1}\int_{2^kR}^{2^{k+1}R}\frac{\rho(t)}{t}dt = CMf(x)\int_0^R\frac{\rho(t)}{t}dt.\end{aligned}$$

Here we have used the fact that

$$\int_{2^kR}^{2^{k+1}R}\frac{\rho(t)}{t}dt \ge C\rho(2^kR)\int_{2^kR}^{2^{k+1}R}\frac{1}{t}dt = C\rho(2^kR)\ln 2. \tag{6.6}$$

Now, using (6.4), we obtain

$$|T_1(x)| \le CMf(x)(\phi(R))^{(p-q)/q}. \tag{6.7}$$

For $T_2(x)$, using (3.3) we have

$$\begin{aligned}|T_2(x)| &\le \int_{|xy^{-1}|\ge R}\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}|f(y)|dy \\ &= \sum_{k=0}^{\infty}\int_{2^kR\le|xy^{-1}|<2^{k+1}R}\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}|f(y)|dy \\ &\le C\sum_{k=0}^{\infty}\frac{\rho(2^kR)}{(2^kR)^Q}\int_{|xy^{-1}|<2^{k+1}R}|f(y)|dy.\end{aligned}$$

From this using Hölder's inequality, we obtain

$$
\begin{aligned}
|T_2(x)| &\leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^Q} \left(\int_{|xy^{-1}|<2^{k+1}R} dy \right)^{1-1/p} \left(\int_{|xy^{-1}|<2^{k+1}R} |f(y)| dy \right)^{1/p} \\
&\leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^{Q/p}} \left(\int_{|xy^{-1}|<2^{k+1}R} |f(y)| dy \right)^{1/p} \\
&\leq C \|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \rho(2^{k+1}R) \phi(2^{k+1}R) \\
&\leq C \|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \int_{2^k R}^{2^{k+1}R} \frac{\rho(t)\phi(t)}{t} = C \|f\|_{LM^{p,\phi}(\mathbb{G})} \int_R^{\infty} \frac{\rho(t)\phi(t)}{t},
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
\int_{2^k R}^{2^{k+1}R} \frac{\rho(t)\phi(t)}{t} dt &\geq C\rho(2^{k+1}R)\phi(2^{k+1}R) \int_{2^k R}^{2^{k+1}R} \frac{1}{t} dt \\
&= C\rho(2^{k+1}R)\phi(2^{k+1}R) \ln 2.
\end{aligned} \tag{6.8}
$$

Now, in view of (6.4), we obtain

$$
|T_2(x)| \leq C \|f\|_{LM^{p,\phi}(\mathbb{G})} (\phi(R))^{p/q}. \tag{6.9}
$$

Summing the two estimates (6.7) and (6.9), we arrive at

$$
|T_\rho f(x)| \leq C(Mf(x)(\phi(R))^{(p-q)/q} + \|f\|_{LM^{p,\phi}(\mathbb{G})} (\phi(R))^{p/q}).
$$

Assuming that f is not identically 0 and that Mf is finite everywhere and then using the fact that ϕ is surjective, we can choose $R > 0$ such that $\phi(R) = Mf(x) \cdot \|f\|^{-1}_{LM^{p,\phi}(\mathbb{G})}$. Thus, for every $x \in \mathbb{G}$, we have

$$
|T_\rho f(x)| \leq C(Mf(x))^{\frac{p}{q}} \|f\|^{\frac{q-p}{q}}_{LM^{p,\phi}(\mathbb{G})}.
$$

It follows that

$$
\left(\int_{B(0,r)} |T_\rho f(x)|^q \right)^{1/q} \leq C \left(\int_{B(0,r)} |Mf(x)|^p \right)^{1/q} \|f\|^{\frac{q-p}{q}}_{LM^{p,\phi}(\mathbb{G})},
$$

then we divide both sides by $(\phi(r))^{p/q} r^{Q/q}$ to get

$$
\begin{aligned}
&\frac{1}{(\phi(r))^{p/q}} \left(\frac{1}{r^Q} \int_{B(0,r)} |T_\rho f(x)|^q \right)^{1/q} \\
&\qquad \leq C \frac{1}{(\phi(r))^{p/q}} \left(\frac{1}{r^Q} \int_{B(0,r)} |Mf(x)|^p \right)^{1/q} \|f\|^{\frac{q-p}{q}}_{LM^{p,\phi}(\mathbb{G})}.
\end{aligned}
$$

Taking the supremum over $r>0$ and using the boundedness of the maximal operator M on $LM^{p,\phi}(\mathbb{G})$ (3.6), we obtain

$$\|T_\rho f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})} \leq C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})}.$$

The proof is complete. □

Now let us turn to the Olsen type inequalities for the generalized fractional integral operator T_ρ and Bessel-Riesz operator $I_{\alpha,\gamma}$.

THEOREM 6.2. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm. Let ρ and ϕ satisfy the doubling condition* (3.3). *Let ϕ be also surjective and satisfy* (6.3)-(6.4). *Then we have*

$$\|WT_\rho f\|_{LM^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}, \quad 1<p<p_2<\infty, \tag{6.10}$$

provided that $W \in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$.

Proof of Theorem 6.2. By using Hölder's inequality, we have

$$\frac{1}{r^Q}\int_{B(0,r)}|W(x)T_\rho f(x)|^p dx$$
$$\leq \left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{p/p_2}\left(\frac{1}{r^Q}\int_{B(0,r)}|T_\rho f(x)|^{\frac{pp_2}{p_2-p}}dx\right)^{\frac{p_2-p}{p_2}}.$$

Now let us take the p-th roots and then divide both sides by $\phi(r)$ to obtain

$$\frac{1}{\phi(r)}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)T_\rho f(x)|^p dx\right)^{1/p}$$
$$\leq \frac{1}{(\phi(r))^{p/p_2}}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{1/p_2}$$
$$\times \frac{1}{(\phi(r))^{\frac{p_2-p}{p_2}}}\left(\frac{1}{r^Q}\int_{B(0,r)}|T_\rho f(x)|^{\frac{pp_2}{p_2-p}}dx\right)^{\frac{p_2-p}{pp_2}}.$$

By taking the supremum over $r>0$ and using the inequality (6.5), we get

$$\|WT_\rho f\|_{LM^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|T_\rho f\|_{L^{\frac{pp_2}{p_2-p},\phi^{\frac{p_2-p}{p_2}}}(\mathbb{G})}.$$

Taking into account that $1<p<\frac{pp_2}{p_2-p}<\infty$ and putting $q=\frac{pp_2}{p_2-p}$ in (6.5), we obtain (6.10). □

THEOREM 6.3. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the doubling condition and assume that $\omega(r) \leq Cr^{-\alpha}$ for every $r > 0$, so that $K_{\alpha,\gamma} \in LM^{p_2,\omega}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha} < p_2 < \frac{Q}{Q-\alpha}$ and $p_2 \geq 1$, where $0 < \alpha < Q$, $1 < p < \infty, q = \frac{\beta p}{\beta+Q-\alpha}$ and $\gamma > 0$. If $\phi(r) \leq Cr^\beta$ for every $r > 0$, where $\beta < -\alpha < -Q - \beta$, then we have*

$$\|WI_{\alpha,\gamma}f\|_{LM^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{6.11}$$

provided that $W \in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$, where $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{q}$.

Proof of Theorem 6.3. As in Theorem 6.2, by using Hölder's inequality for $\frac{p}{p_2} + \frac{p}{q} = 1$, we have

$$\begin{aligned}\frac{1}{r^Q}\int_{B(0,r)} &|W(x)I_{\alpha,\gamma}f(x)|^p dx \\ &\leq \left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{p/p_2}\left(\frac{1}{r^Q}\int_{B(0,r)}|I_{\alpha,\gamma}f(x)|^q dx\right)^{p/q}.\end{aligned}$$

Now we take the p-th roots and then divide both sides by $\phi(r)$ to get

$$\begin{aligned}\frac{1}{\phi(r)}&\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)I_{\alpha,\gamma}f(x)|^p dx\right)^{1/p} \\ &\leq \frac{1}{(\phi(r))^{p/p_2}}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{1/p_2} \\ &\qquad \times \frac{1}{(\phi(r))^{p/q}}\left(\frac{1}{r^Q}\int_{B(0,r)}|I_{\alpha,\gamma}f(x)|^q dx\right)^{1/q}.\end{aligned}$$

By taking the supremum over $r > 0$, we have

$$\|WI_{\alpha,\gamma}f\|_{LM^{p,\phi}(\mathbb{G})} \leq C\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|I_{\alpha,\gamma}f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})},$$

which implies (6.11) in view of Theorem 4.3 after putting $\psi(r) = (\phi(r))^{p/q}$. $\square$

7. Inequalities for the modified version of generalized fractional integral operator in Campanato spaces

In this section, we prove the boundedness of the modified version of the operator T_ρ in Campanato spaces on homogeneous groups.

We define the generalized local (central) Campanato space by

$$\mathcal{LM}^{p,\phi}(\mathbb{G}) := \{f \in L^p_{loc}(\mathbb{G}) : \|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})} < \infty\}, \tag{7.1}$$

where

$$\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})} := \sup_{r>0} \frac{1}{\phi(r)} \left(\frac{1}{r^Q} \int_{B(0,r)} |f(x) - f_B|^p dx \right)^{1/p},$$

with $f_B = f_{B(0,r)} := \frac{1}{r^Q} \int_{B(0,r)} f(y)dy$, and we assume that $\frac{\phi(r)}{r}$ is nonincreasing.

Next, for the function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, we define the modified version of the generalized fractional integral operator T_ρ by

$$\widetilde{T_\rho} f(x) := \int_{\mathbb{G}} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)(1 - \chi_{B(0,1)}(y))}{|y|^Q} \right) f(y)dy, \tag{7.2}$$

where $B(0,1) := \{x \in \mathbb{G} : |x| < 1\}$ and $\chi_{B(0,1)}$ is the characteristic function of $B(0,1)$. In this definition, we assume that ρ satisfies (6.2), (3.3) and the following conditions:

$$\int_r^\infty \frac{\rho(t)}{t^2} dt \le C_1 \frac{\rho(r)}{r} \quad \text{for all} \quad r > 0; \tag{7.3}$$

$$\frac{1}{2} \le \frac{r}{s} \le 2 \Rightarrow \left| \frac{\rho(r)}{r^Q} - \frac{\rho(s)}{s^Q} \right| \le C_2 |r - s| \frac{\rho(s)}{s^{Q+1}}. \tag{7.4}$$

For instance, the function $\rho(r) = r^\alpha$ satisfies (6.2), (3.3) and (7.4) for $0 < \alpha < Q$, and also satisfies (7.3) for $0 < \alpha < 1$.

THEOREM 7.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm. Let ρ satisfy* (6.2), (3.3), (7.3), (7.4), *and let ϕ satisfy the doubling condition* (3.3) *and $\int_1^\infty \frac{\phi(t)}{t} dt < \infty$. If*

$$\int_r^\infty \frac{\phi(t)}{t} dt \int_0^r \frac{\rho(t)}{t} dt + r \int_r^\infty \frac{\rho(t)\phi(t)}{t^2} dt \le C_3 \psi(r) \, \text{for all} \, r > 0, \tag{7.5}$$

then we have

$$\|\widetilde{T}_\rho f\|_{\mathcal{LM}^{p,\psi}(\mathbb{G})} \le C_{p,\phi,Q} \|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}, \quad 1 < p < \infty. \tag{7.6}$$

Proof of Theorem 7.1. For every $x \in B(0,r)$ and $f \in \mathcal{LM}^{p,\phi}(\mathbb{G})$, let us write $\widetilde{T}_\rho f$ in the following form:

$$\widetilde{T}_\rho f(x) = \widetilde{T}_{B(0,r)}(x) + C^1_{B(0,r)} + C^2_{B(0,r)} = \widetilde{T}^1_{B(0,r)}(x) + \widetilde{T}^2_{B(0,r)}(x) + C^1_{B(0,r)} + C^2_{B(0,r)},$$

where

$$\widetilde{T}_{B(0,r)}(x) := \int_{\mathbb{G}} (f(y) - f_{B(0,2r)}) \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)(1 - \chi_{B(0,2r)}(y))}{|y|^Q} \right) dy,$$

$$C^1_{B(0,r)} := \int_{\mathbb{G}} (f(y) - f_{B(0,2r)}) \times \left(\frac{\rho(|y|)(1-\chi_{B(0,2r)}(y))}{|y|^Q} - \frac{\rho(|y|)(1-\chi_{B(0,1)}(y))}{|y|^Q} \right) dy,$$

$$C^2_{B(0,r)} := \int_{\mathbb{G}} f_{B(0,2r)} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)(1-\chi_{B(0,1)}(y))}{|y|^Q} \right) dy,$$

$$\widetilde{T}^1_{B(0,r)}(x) := \int_{B(0,2r)} (f(y) - f_{B(0,2r)}) \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} dy,$$

$$\widetilde{T}^2_{B(0,r)}(x) := \int_{B^c(0,2r)} (f(y) - f_{B(0,2r)}) \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q} \right) dy.$$

Since

$$\left| \frac{\rho(|y|)(1-\chi_{B(0,2r)}(y))}{|y|^Q} - \frac{\rho(|y|)(1-\chi_{B(0,1)}(y))}{|y|^Q} \right| \leq \begin{cases} 0, |y| < \min(1,2r) \text{ or } |y| \geq \max(1,2r); \\ \frac{\rho(|y|)}{|y|^Q} = \text{const}, \text{ otherwise}, \end{cases}$$

$C^1_{B(0,r)}$ is finite.

Now let us show that $C^2_{B(0,r)}$ is finite. For this it is enough to prove that the following integral is finite:

$$\int_{\mathbb{G}} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)(1-\chi_{B(0,1)}(y))}{|y|^Q} \right) dy = \int_{\mathbb{G}} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q} \right) dy + \int_{B(0,1)} \frac{\rho(|y|)}{|y|^Q} dy.$$

Let us denote $A := \int_{\mathbb{G}} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q} \right) dy$. For large $R > 0$, we write A in the form $A = A_1 + A_2 + A_3$, where

$$A_1 := \int_{B(x,R)} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} dy - \int_{B(0,R)} \frac{\rho(|y|)}{|y|^Q} dy,$$

$$A_2 := \int_{B(x,R+r)\setminus B(x,R)} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} dy - \int_{B(x,R+r)\setminus B(0,R)} \frac{\rho(|y|)}{|y|^Q} dy,$$

$$A_3 := \int_{B^c(x,R+r)} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q} \right) dy.$$

Since we have $\int_0^1 \frac{\rho(t)}{t} dt < +\infty$, it implies that $\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}, \frac{\rho(|y|)}{|y|^Q} \in L^1_{loc}(\mathbb{G})$, and hence $A_1 = 0$. By (7.4) we have

$$
\begin{aligned}
A_3 &\leq \int_{B^c(x,R+r)} \left| \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q} \right| dy \\
&\leq C \int_{B^c(x,R+r)} ||xy^{-1}| - |y|| \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^{Q+1}} dy.
\end{aligned}
$$

By using the triangle inequality (see e.g. [14, Theorem 3.1.39, p.113]) and symmetric property of homogeneous quasi-norms, we get

$$
\begin{aligned}
A_3 &\leq C||x| + |y^{-1}| - |y|| \int_{R+r}^{+\infty} \int_{\mathfrak{S}} \frac{\rho(t)}{t^{Q+1}} t^{Q-1} d\sigma(y) dt \\
&\leq C|\sigma| r \int_{R+r}^{+\infty} \frac{\rho(t)}{t^2} dt.
\end{aligned}
$$

The inequality (7.3) implies that the last integral is finite and $|A_3| \to 0$ as $R \to +\infty$. For A_2, we have

$$
\begin{aligned}
|A_2| &\leq \int_{B(x,R+r)\setminus B(x,R-r)} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} + \frac{\rho(|y|)}{|y|^Q} \right) dy \\
&\sim ((R+r)^Q - (R-r)^Q) \frac{\rho(R)}{R^Q} \leq Cr \frac{\rho(R)}{R},
\end{aligned}
$$

and taking into account the conditions (3.3) and (7.3), we obtain

$$
|A_2| \leq Cr \frac{\rho(R)}{R} \to 0 \text{ as } R \to +\infty.
$$

Since $A \to 0$ as $R \to +\infty$, we have $A = 0$ and hence

$$
\int_{\mathbb{G}} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)(1 - \chi_{B(0,1)}(y))}{|y|^Q} \right) dy = \int_{B(0,1)} \frac{\rho(|y|)}{|y|^Q} dy < \infty,
$$

which implies that $C^2_{B(0,r)}$ is finite.

Now before estimating $\widetilde{T}^1_{B(0,r)}$, let us denote $\widetilde{f} := (f - f_{B(0,2r)})\chi_{B(0,2r)}$ and $\widetilde{\phi}(r) := \int_r^\infty \frac{\phi(t)}{t} dt$. Then, we have

$$
\begin{aligned}
|\widetilde{T}^1_{B(0,r)}(x)| &\leq \int_{B(0,2r)} |\widetilde{f}(y)| \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} dy \\
&= \sum_{k=-\infty}^{0} \int_{2^k r \leq |xy^{-1}| < 2^{k+1} r} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} |\widetilde{f}(y)| dy.
\end{aligned}
$$

By using (3.3) and (6.6), we get

$$
\begin{aligned}
|\widetilde{T}^1_{B(0,r)}(x)| &\le C \sum_{k=-\infty}^{0} \frac{\rho(2^k r)}{(2^k r)^Q} \int_{|xy^{-1}|<2^{k+1}r} |\widetilde{f}(y)|dy \\
&\le CM\widetilde{f}(x) \sum_{k=-\infty}^{0} \rho(2^k r) \le CM\widetilde{f}(x) \sum_{k=-\infty}^{0} \rho(2^{k-1} r) \\
&\le CM\widetilde{f}(x) \sum_{k=-\infty}^{0} \int_{2^{k-1}r}^{2^k r} \frac{\rho(t)}{t} dt = CM\widetilde{f}(x) \int_0^r \frac{\rho(t)}{t} dt.
\end{aligned}
$$

Now using (7.5), we have $|\widetilde{T}^1_{B(0,r)}(x)| \le C\frac{\psi(r)}{\widetilde{\phi}(r)} M\widetilde{f}(x)$. It follows that

$$
\begin{aligned}
\frac{1}{\psi(r)}\left(\frac{1}{r^Q}\int_{B(0,r)} |\widetilde{T}^1_{B(0,r)}(x)|^p dx\right)^{1/p} &\le C\frac{1}{\widetilde{\phi}(r) r^{Q/p}}\left(\int_{B(0,r)} |M\widetilde{f}(x)|^p dx\right)^{1/p} \\
&\le C\frac{1}{\widetilde{\phi}(r) r^{Q/p}} \|\widetilde{f}\|_{L^p(\mathbb{G})},
\end{aligned}
$$

where we have used (3.8).

By the Minkowski inequality, we have

$$
\begin{aligned}
&\frac{1}{\widetilde{\phi}(r) r^{Q/p}} \|\widetilde{f}\|_{L^p(\mathbb{G})} = \frac{1}{\widetilde{\phi}(r) r^{Q/p}} \|(f - f_{B(0,2r)})\chi_{B(0,2r)}\|_{L^p(\mathbb{G})} \\
&\le C\frac{1}{\widetilde{\phi}(r) r^{Q/p}} (\|(f-\sigma(f))_{\chi_{B(0,2r)}}\|_{L^p(\mathbb{G})} + (2r)^{Q/p}|f_{B(0,2r)} - \sigma(f)|),
\end{aligned}
$$

where $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)}$.

Moreover, we obtain the following inequalities exactly in the same way as in the Abelian case (see [13], Section 6)

$$
\|f-\sigma(f)\|_{LM^{p,\widetilde{\phi}}(\mathbb{G})} \le C_1 \|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}, \tag{7.7}
$$

and

$$
|f_{B(0,r)} - \sigma(f)| \le C_2 \|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})} \widetilde{\phi}(r). \tag{7.8}
$$

Finally, using these inequalities we get our estimate for $\widetilde{T}^1_{B(0,r)}$ as

$$
|\widetilde{T}^1_{B(0,r)}(x)| \le C\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}. \tag{7.9}
$$

Now let us estimate $\widetilde{T}^2_{B(0,r)}$. By (3.3) and (7.4), we have

$$
\begin{aligned}
|\widetilde{T}^2_{B(0,r)}(x)| &\le \int_{B^c(0,2r)} |f(y) - f_{B(0,2r)}| \left| \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q} \right| dy \\
&\le C||xy^{-1}| - |y|| \int_{|y|\ge 2r} |f(y) - f_{B(0,2r)}| \frac{\rho(|y|)}{|y|^{Q+1}} dy.
\end{aligned}
$$

By using the triangle inequality (see e.g. [14, Theorem 3.1.39, p.113]) and symmetric property of homogeneous quasi-norms, we get

$$
\begin{aligned}
|\widetilde{T}^2_{B(0,r)}(x)| &\le C||x|+|y^{-1}|-|y||\int_{|y|\ge 2r}|f(y)-f_{B(0,2r)}|\frac{\rho(|y|)}{|y|^{Q+1}}dy\\
&\le C|x|\int_{|y|\ge 2r}|f(y)-f_{B(0,2r)}|\frac{\rho(|y|)}{|y|^{Q+1}}dy\\
&= C|x|\sum_{k=2}^{\infty}\int_{2^{k-1}r\le|y|<2^k r}\frac{\rho(|y|)|f(y)-f_{B(0,2r)}|}{|y|^{Q+1}}dy.
\end{aligned}
$$

By using (3.3) and Hölder's inequality, we obtain

$$
|\widetilde{T}^2_{B(0,r)}(x)| \le C|x|\sum_{k=2}^{\infty}\frac{\rho(2^k r)}{(2^k r)^{Q+1}}\int_{|y|<2^k r}|f(y)-f_{B(0,2r)}|dy
$$

$$
\le C|x|\sum_{k=2}^{\infty}\frac{\rho(2^k r)}{2^k r}\left(\frac{1}{(2^k r)^Q}\int_{|y|<2^k r}|f(y)-f_{B(0,2r)}|^p dy\right)^{1/p}.
$$

As in the Abelian case ([13]), we have

$$
\left(\frac{1}{(2^k r)^Q}\int_{B(0,2^k r)}|f(y)-f_{B(0,2r)}|^p dy\right)^{1/p} \le C\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}\int_{2r}^{2^{k+1}r}\frac{\phi(s)}{s}ds,
$$

for every $k\ge 2$. The inequality (6.6) implies that

$$
\int_{2^k r}^{2^{k+1}r}\frac{\rho(t)}{t^2}dt \ge \frac{1}{2^{k+1}r}\int_{2^k r}^{2^{k+1}r}\frac{\rho(t)}{t}dt \ge C\frac{\rho(2^k r)}{2^k r}.
$$

By using the last two inequalities, we get

$$
\begin{aligned}
|\widetilde{T}^2_{B(0,r)}(x)| &\le C|x|\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}\sum_{k=2}^{\infty}\frac{\rho(2^k r)}{2^k r}\int_{2r}^{2^{k+1}r}\frac{\phi(s)}{s}ds\\
&\le C|x|\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}\sum_{k=2}^{\infty}\int_{2^k r}^{2^{k+1}r}\frac{\rho(t)}{t^2}\left(\int_{2r}^{t}\frac{\phi(s)}{s}ds\right)dt\\
&\le C|x|\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}\int_{2r}^{\infty}\frac{\rho(t)}{t^2}\left(\int_{2r}^{t}\frac{\phi(s)}{s}ds\right)dt\\
&= C|x|\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}\int_{2r}^{\infty}\left(\int_{s}^{\infty}\frac{\rho(t)}{t^2}dt\right)\frac{\phi(s)}{s}ds.
\end{aligned}
$$

Using (7.3) and then (7.5), it implies that

$$
|\widetilde{T}^2_{B(0,r)}(x)| \le Cr\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}\int_{2r}^{\infty}\frac{\rho(s)\phi(s)}{s^2}ds \le C\psi(r)\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}.
$$

This follows that

$$\frac{1}{\psi(r)}\left(\frac{1}{r^Q}\int_{B(0,r)}|\widetilde{T}^2_{B(0,r)}(x)|^p dx\right)^{1/p} \leq C\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}. \tag{7.10}$$

Summing the estimates (7.9) and (7.10), we obtain (7.6). □

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References

[1] D. R. Adams, A note on Riesz potentials. *Duke Math. J.* **42** (1975), 765–778.

[2] A. Alsaedi, B. Ahmad and M. Kirane, A survey of useful inequalities in fractional calculus. *Fract. Calc. Appl. Anal.* **20**, No 3 (2017), 574–594; DOI: 10.1515/fca-2017-0031; https://www.degruyter.com/view/j/fca.2017.20.issue-3/issue-files/fca.2017.20.issue-3.xml.

[3] A. Akbulut, V. S. Guliyev and Sh. A. Muradova, Boundedness of the anisotropic Riesz potential in anisotropic local Morrey-type spaces. *Complex Var. Elliptic Equ.* **58**, No 2 (2013), 259–280.

[4] A. Akbulut, I. Ekincioglu, A. Serbetci and T. Tararykova, Boundedness of the anisotropic fractional maximal operator in anisotropic local Morrey-type spaces. *Eurasian Math. J.* **2**, No 2 (2011), 5–30.

[5] V. I. Burenkov, H. V. Guliyev and V. S. Guliyev, Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces. *J. Comput. Appl. Math.* **208**, No 1 (2007), 280–301.

[6] V. I. Burenkov, Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. I, II. *Eurasian Math. J.* **3**, (2012), 11–32, **4** (2013), 21–45.

[7] V. I. Burenkov, D. K. Darbayeva and E. D. Nursultanov, Description of interpolation spaces for general local Morrey-type spaces. *Eurasian Math. J.* **4** (2013), 46–53.

[8] V. I. Burenkov, E. D. Nursultanov and D. K. Chigambayeva, Description of the interpolation spaces for a pair of local Morrey-type spaces and their generalizations. *Proc. Steklov Inst. Math.* **284** (2014), 97–128.

[9] E. Burtseva and N. Samko, Weighted Adams type theorem for the Riesz fractional integral in generalized Morrey space. *Fract. Calc. Appl. Anal.* **19**, No 4 (2016), 954–972; DOI: 10.1515/fca-2016-0052;

https://www.degruyter.com/view/j/fca.2016.19.issue-4/issue-files/fca.2016.19.issue-4.xml.

[10] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function. *Rend. Mat.* **7** (1987), 273–279.

[11] A. Eridani, On the boundedness of a generalized fractional integral on generalized Morrey spaces. *Tamkang J. Math.* **33**, No 4 (2002), 335–340.

[12] A. Eridani and H. Gunawan, On generalized fractional integrals. *J. Indonesian Math. Soc. (MIHMI)* **8**, No 3 (2002), 25–28.

[13] A. Eridani, H. Gunawan and E. Nakai, On generalized fractional integral operators. *Sci. Mathematicae Japanicae Online* **10** (2004), 307–318.

[14] V. Fischer and M. Ruzhansky, *Quantization on Nilpotent Lie Groups.* Ser. Progress in Mathematics, Vol. 314, Birkhäuser (2016).

[15] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups.* Ser. Mathematical Notes, vol. 28, Princeton Univ. Press, Princeton, N.J. & Univ. of Tokyo Press, Tokyo (1982).

[16] V. S. Guliyev, Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{G}$. *Doctor's Degree Dissertation*, Moscow, Mat. Inst. Steklov (1994), 1–329.

[17] V. S. Guliyev, Local generalized Morrey spaces and singular integrals with rough kernel. *Azerb. J. Math.* **3**, No 2 (2013), 79–94.

[18] V. S. Guliyev, Generalized local Morrey spaces and fractional integral operators with rough kernel. Problems in mathematical analysis. *J. Math. Sci. (N.Y.)* **193**, No 2 (2013), 211–227.

[19] V. Guliyev, A. Akbulut and Y. Mammadov, Boundedness of fractional maximal operator and their higher order commutators in generalized Morrey spaces on Carnot groups. *Acta Math. Sci. Ser. B Engl. Ed.* **33**, No 5 (2013), 1329–1346.

[20] V. Guliyev and L. Softova, Generalized Morrey estimates for the gradient of divergence form parabolic operators with discontinuous coefficients. *J. Differential Equations* **259** (2015), 2368–2387.

[21] V. Guliyev and L. Softova, Generalized Morrey regularity for parabolic equations with discontinuous data. *Proc. Edinb. Math. Soc.* **58** (2015), 199–218.

[22] V. Guliyev and S. Samko, Maximal operator in variable exponent generalized Morrey spaces on quasi-metric measure space. *Mediterr. J. Math.* **13** (2016), 1151–1165.

[23] H. Gunawan and A. Eridani, Fractional integrals and generalized Olsen inequalities. *Kyungpook Math. J.* **49** (2009), 31–39.

[24] G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals. I. *Math. Zeit.* **27** (1927), 565–606.

[25] G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals. II. *Math. Zeit.* **34** (1932), 403–439.

[26] M. Idris, H. Gunawan, J. Lindiarni and A. Eridani, The boundedness of Bessel-Riesz operators on Morrey spaces. *AIP Conf. Proc.* **1729** (2015); DOI: 10.1063/1.4946909.

[27] M. Idris, H. Gunawan and A. Eridani, The boundedness of Bessel-Riesz operators on generalized Morrey spaces. *Aust. J. Math. Anal. Appl.* **13**, No 1 (2016), 1–10.

[28] K. Kurata, S. Nishigaki and S. Sugano, Boundedness of integral operator on generalized Morrey spaces and its application to Schrödinger operator. *Proc. Amer. Math. Soc.* **128** (1999), 587–602.

[29] E. Nakai, Hardy-Littlewood maximal operator, singular integral operator and Riesz potentials on generalized Morrey spaces. *Math. Nachr.* **166** (1994), 95–103.

[30] E. Nakai, On generalized fractional integrals. *Taiwanese J. Math.* **5** (2001), 587–602.

[31] E. Nakai, On generalized fractional integrals on the weak Orlicz spaces, BMO_ϕ, the Morrey spaces and the Campanato spaces. In: *Function Spaces, Interpolation Theory and Related Topics (Lund, 2000)* (2002), De Gruyter, Berlin, 389–401.

[32] M. Ruzhansky and D. Suragan, Local Hardy and Rellich inequalities for sums of squares of vector fields. *Adv. Diff. Equations* **22**, No 7-8 (2017), 505–540.

[33] M. Ruzhansky and D. Suragan, On horizontal Hardy, Rellich, Caffarelli-Kohn-Nirenberg and p-sub-Laplacian inequalities on stratified groups. *J. Differential Equations* **262** (2017), 1799–1821.

[34] M. Ruzhansky and D. Suragan, Anisotropic L^2-weighted Hardy and L^2-Caffarelli-Kohn-Nirenberg inequalities. *Commun. Contemp. Math.* **19**, No 6 (2017); DOI: 10.1142/S0219199717500146.

[35] M. Ruzhansky and D. Suragan, Layer potentials, Green formulae, Kac problem, and refined Hardy inequality on homogeneous Carnot groups. *Adv. Math.* **308**, (2017), 483–528.

[36] M. Ruzhansky and D. Suragan, Hardy and Rellich inequalities, identities, and sharp remainders on homogeneous groups. *Adv. Math.* **317**, (2017), 799–822.

[37] H. Rafeiro, N. Samko and S. Samko, Morrey-Campanato Spaces: an Overview. *Oper. Theory Adv. Appl.* **228** (2013), 293–323.

[38] M. Ruzhansky, D. Suragan and N. Yessirkegenov, Extended Caffarelli-Kohn-Nirenberg inequalities and superweights for L^p-weighted Hardy inequalities. *C. R. Math. Acad. Sci. Paris* **355**, No 6 (2017), 694–698.

[39] M. Ruzhansky, D. Suragan and N. Yessirkegenov, Caffarelli-Kohn-Nirenberg and Sobolev type inequalities on stratified Lie groups. *NoDEA Nonlinear Differential Equations Appl.* **24** (2017); DOI: 10.1007/s00030-017-0478-2.

[40] M. Ruzhansky, D. Suragan and N. Yessirkegenov, Sobolev type inequalities, Euler-Hilbert-Sobolev and Sobolev-Lorentz-Zygmund spaces on homogeneous groups. *Integral Equations and Operator Theory* **90** (2018); DOI: 10.1007/s00020-018-2437-7.

[41] M. Ruzhansky, D. Suragan and N. Yessirkegenov, Extended Caffarelli-Kohn-Nirenberg inequalities, and remainders, stability, and superweights for L_p-weighted Hardy inequalities. *Trans. Amer. Math. Soc. Ser. B* **5** (2018), 32–62.

[42] Y. Sawano, S. Sugano and H. Tanaka, Orlicz-Morrey spaces and fractional operators. *Potential Anal.* **36** (2012), 517–556.

[43] S. L. Sobolev, On a theorem in functional analysis (In Russian). *Math. Sbor.* **46** (1938), 471–497 [English transl. in: *Amer. Math. Soc. Transl.* **34**, No 2 (1963), 39–68].

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